MATH 303 – Measures and Integration
Homework 12

Problem 1. Let $(X, \mathcal{B})$ be a measurable space.

(a) Let $\mu, \nu, \rho : \mathcal{B} \to [0, \infty]$ be σ-finite measures, and suppose $\rho \ll \nu \ll \mu$. Prove the "chain rule"

$$\frac{d\rho}{d\mu} = \frac{d\rho}{d\nu}\frac{d\nu}{d\mu}.$$

(b) Suppose μ, ν are finite (positive) measures on $(X, \mathcal{B})$, and $\mu \approx \nu$. Show that the Radon–Nikodym derivative $f = \frac{d\nu}{d\mu}$ satisfies $0 < f < \infty$ μ-a.e., and $\frac{d\mu}{d\nu} = \frac{1}{f}$ ν-a.e.

Solution: **(a)** Let $f = \frac{d\rho}{d\nu}$ and $g = \frac{d\nu}{d\mu}$. We want to show $\frac{d\rho}{d\mu} = fg$. Let $E \in \mathcal{B}$. Then

$$\rho(E) \overset{(*)}{=} \int_E f \; d\nu \overset{(**)}{=} \int_E fg \; d\mu,$$

where in the equality $(*)$ we have used the definition of the Radon–Nikodym derivative $\frac{d\rho}{d\nu}$ and in $(**)$ we have used Proposition 10.16. Thus, $d\rho = fg \; d\mu$ as desired.

(b) Let $f = \frac{d\nu}{d\mu}$. Then by (a), $1 = \frac{d\nu}{d\nu} = f \cdot \frac{d\mu}{d\nu}$. In order for this product of two nonnegative extended real numbers to be 1, we must have $0 < f < \infty$ a.e. and $\frac{d\mu}{d\nu} = \frac{1}{f}$ a.e.

Problem 2. Let $(X, \mathcal{B})$ be a measurable space.

(a) Let $\mu : \mathcal{B} \to [-\infty, \infty]$ be a signed measure, and let $|\mu|$ be the total variation measure. Show that $\mu \ll |\mu|$ and describe the Radon–Nikodym derivative $\frac{d\mu}{d|\mu|}$.

(b) Suppose $\mu : \mathcal{B} \to \mathbb{C}$ is a complex measure. Define the *total variation measure* $|\mu|$ by

$$|\mu|(E) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : E = \bigsqcup_{n \in \mathbb{N}} E_n \right\}$$

for $E \in \mathcal{B}$. Suppose ν is a σ-finite positive measure on $(X, \mathcal{B})$ such that $\mu \ll \nu$. (For example, applying the Jordan decomposition theorem to the real and imaginary parts of μ and writing $\mu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ as a combination of positive finite measures, one can take $\nu = \sum_{j=1}^4 \nu_j$.) Prove that $|\mu|$ is a measure and $\frac{d|\mu|}{d\nu} = \left|\frac{d\mu}{d\nu}\right|$ a.e. Conclude that there exists a measurable function $\theta : X \to [0, 1)$ such that $\frac{d\mu}{d|\mu|}(x) = e^{2\pi i \theta(x)}$ for $|\mu|$-a.e. $x \in X$.

Solution: **(a)** Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ so that $|\mu| = \mu^+ + \mu^-$. If $|\mu|(0)$, then $\mu^+(0) = \mu^-(0) = 0$, so $\mu(0) = 0$. Therefore, $\mu \ll |\mu|$.

We claim that the Radon–Nikodym derivative of μ with respect to $|\mu|$ is given by

$$\frac{d\mu}{d|\mu|} = \mathbb{1}_P - \mathbb{1}_N,$$

where (P,N) is a Hahn decomposition of μ. Indeed, for any $E \in \mathcal{B}$,

$$\mu(E) = \mu(E \cap P) + \mu(E \cap N) = |\mu|(E \cap P) - |\mu|(E \cap N) = \int_E (\mathbb{1}_P - \mathbb{1}_N)\ d|\mu|.$$

(b) Let $f = \frac{d\mu}{d\nu}$. We will show

$$|\mu|(E) = \int_E |f|\ d\nu$$

for $E \in \mathcal{B}$, which establishes simultaneously that $|\mu|$ is a measure and that $\frac{d|\mu|}{d\nu} = \left|\frac{d\mu}{d\nu}\right|$ a.e.

Suppose $E = \bigsqcup_{n \in \mathbb{N}} E_n$. Then by the triangle inequality for integrals and Theorem 3.12,

$$\sum_{n=1}^{\infty} |\mu(E_n)| = \sum_{n=1}^{\infty} \left| \int_{E_n} f\ d\nu \right| \leq \sum_{n=1}^{\infty} \int_{E_n} |f|\ d\nu = \int_E |f|\ d\nu.$$

Thus, taking a supremum over all countable measurable partitions of E, we have

$$\int_E |f|\ d\nu \geq |\mu|(E).$$

To prove the reverse inequality, we will partition E into portions where the values of f belong to a thin radial slice. Let $\Theta : \mathbb{C} \to [0,1)$ be the function such that $\Theta(0) = 0$ and $z = |z|e^{2\pi i \Theta(z)}$ for $z \neq 0$. Fix $N \in \mathbb{N}$, and partition $\mathbb{C}$ into N radial slices $I_{N,n} = \Theta^{-1}\left(\left[\frac{n}{N}, \frac{n+1}{N}\right)\right)$ for $0 \leq n \leq N-1$. If $\alpha = e^{-2\pi i \theta}$ with $\frac{n}{N} \leq \theta < \frac{n+1}{N}$, and $z \in I_{N,n}$, then

$$|\alpha z - |z|| = \left| |z| e^{2\pi i (\Theta(z) - \theta)} - |z| \right| \leq |z| \left| e^{2\pi i/N} - 1 \right| \leq \frac{2\pi}{N}|z|.$$

Let $E_{N,n} = E \cap \{f \in I_{N,n}\}$. Then letting $\alpha_{N,n}$ be such that $\alpha_{N,n} \int_{E \cap \{f \in I_{N,n}\}} f\ d\nu = \left| \int_{E \cap \{f \in I_{N,n}\}} f\ d\nu \right|$, we have

$$\begin{aligned}
\int_E |f|\ d\nu - \sum_{n=0}^{N-1} |\mu(E_{N,n})| &= \sum_{n=0}^{N-1} \left(\int_{E \cap \{f \in I_{N,n}\}} |f|\ d\nu - \left| \int_{E \cap \{f \in I_{N,n}\}} f\ d\nu \right| \right) \\
&= \sum_{n=0}^{N-1} \left(\int_{E \cap \{f \in I_{N,n}\}} |f|\ d\nu - \int_{E \cap \{f \in I_{N,n}\}} \alpha_{N,n} f\ d\nu \right) \\
&\leq \sum_{n=0}^{N-1} \int_{E \cap \{f \in I_{N,n}\}} ||f| - \alpha_{N,n} f|\ d\nu \\
&\leq \sum_{n=0}^{N-1} \int_{E \cap \{f \in I_{N,n}\}} \frac{2\pi}{N} |f|\ d\nu \\
&= \frac{2\pi}{N} \int_E |f|\ d\nu.
\end{aligned}$$

Since $f \in L^1(\nu)$, this quantity tends to zero as $N \to \infty$. Hence, $\int_E |f| \, d\nu \le |\mu|(E)$.

Now that we have established that $|\mu|$ is a measure, we may take $\nu = |\mu|$ to conclude $\left|\frac{d\mu}{d|\mu|}\right| = \frac{d|\mu|}{d|\mu|} = 1$. Then putting $\theta = \Theta \circ \frac{d\mu}{d|\mu|}$, we have $\frac{d\mu}{d|\mu|} = e^{2\pi i\theta}$ a.e.